# On games corresponding to sequencing situations with ready times

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*Published in:*
Mathematical Programming

*Publication date:*
1995

Mathematical Programming 69 (1995) 471–483

# On games corresponding to sequencing situations with ready times

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Received 12 February 1993; revised manuscript received 21 September 1994

**Abstract**

This paper considers the special class of cooperative sequencing games that arise from one-machine sequencing situations in which all jobs have equal processing times and the ready time of each job is a multiple of the processing time.

By establishing relations between optimal orders of subcoalitions, it is shown that each sequencing game within this class is convex.

*Keywords:* Cooperative games; One-machine sequencing problems

## 1. Introduction

In one-machine sequencing situations each agent (player) has one job that has to be processed on a single machine. Each job is specified by its ready time, the earliest time that the processing of the job can begin, and its processing time, the time the machine takes to handle the job. We assume that the costs of a player depend linearly on the completion time of his job. Furthermore, there is an initial order on the jobs of the agents before the processing of the machine starts.

A group of agents (a coalition) can save costs by rearranging their jobs in a way that is admissible with respect to the various ready times and the initial order. By defining the value of a coalition as the maximal cost savings a coalition can make in this way, we obtain a cooperative sequencing game related to a one-machine sequencing situation. The formal model can be found in Section 2.

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[1] This author is financially supported by the Dutch Organization for Scientific Research (NWO).


*SSDI* 0025-5610(94)00074-3

The above game-theoretic approach was initiated in [2]. Convexity was shown for all sequencing games arising from one-machine sequencing situations in which all jobs have zero ready times. In Section 3 it is seen that this convexity result also holds for the special class of sequencing games that arise from one-machine sequencing situations in which all jobs have equal processing times and the ready time of each job is a multiple of the processing time.

There are several arguments to ask for convexity. Convex games are balanced, which means that one can find core elements prescribing an allocation of the worth (in sequencing games the cost savings) of the grand coalition among the players in such a way that no subgroup has an incentive to split off. Moreover, Shapley [8] and Ichiishi [5] (cf. [4]) showed that the extreme points of the core are the marginal vectors of the game if and only if the game is convex. Here, a marginal vector allocates to each player the marginal contribution this player constitutes according to a given way (a permutation) to form the grand coalition. With respect to one-point game-theoretical solution concepts, convex games are nice since the Shapley value [7], which by definition is the average of all marginal vectors, is the barycentre of the core. Further, the $\tau$-value [10], which is an efficient compromise between a utopia vector and a minimal right vector, can be easily calculated. In Section 4 some of the above properties will be illustrated for a sequencing game corresponding to a sequencing situation in which all processing times are equal to one and all ready times are integers.

The Appendix proves the rather technical lemmata needed for the convexity result of Section 3. These lemmata describe relations between optimal orders of various subcoalitions.

## 2. Sequencing situations

In a one-machine sequencing situation there is a queue of agents, each with one job, before a machine (counter). Each agent (player) has to process his job on the machine. The finite set of agents is denoted by $N$ and $|N| = n$. By a bijection $\sigma : N \to \{1, \dots, n\}$ we can describe the position of the agents in the queue. Specifically, $\sigma(i) = j$ means that player $i$ is in position $j$. The ready time $r_i$ of the job of agent $i$ is the earliest time the processing of this job can begin. The processing time $p_i$ of the job of agent $i$ is the time the machine takes to handle this job.

We assume that every agent has an affine cost function $c_i : [0, \infty) \to \mathbb{R}$ defined by $c_i(t) = \alpha_i t + \beta_i$ with $\alpha_i > 0$, $\beta_i \in \mathbb{R}$.

Further it is assumed that there is an initial order $\sigma_0 : N \to \{1, \dots, n\}$ on the jobs of the players before the processing of the machine starts with the property that

$$\text{(A1)} \qquad r_i \leqslant r_j, \quad \text{for all } i, j \in N \text{ with } \sigma_0(i) < \sigma_0(j).$$

In fact, one can view the initial order as being determined by the various time moments the various jobs can enter the system (i.e., join the queue).

A sequencing situation as described above is denoted by $(\sigma_0, r, p, \alpha)$, where $\sigma_0 : N \to \{1, \dots, n\}$, $r = (r_i)_{i \in N} \in [\mathbf{0}, \infty)^N$, $p = (p_i)_{i \in N} \in \mathbb{R}_+^N$ and $\alpha = (\alpha_i)_{i \in N} \in \mathbb{R}_+^N$. The vector $\beta = (\beta_i)_{i \in N} \in \mathbb{R}^N$ of fixed costs for the agents can be omitted in the description of a sequencing situation because, in the sequel, we will focus on cost savings.

For player $i \in N$ we define the following sets with respect to a bijection $\sigma$. The set of predecessors of player $i$ is $P(\sigma, i) := \{j \mid \sigma(j) < \sigma(i)\}$ and the set of followers of player $i$ is $F(\sigma, i) := \{j \mid \sigma(j) > \sigma(i)\}$. For notational convenience let $P(i) := P(\sigma_0, i)$ and $F(i) := F(\sigma_0, i)$.

The starting time $t_{\sigma,i}$ of the job of agent $i$ if processed according to a bijection $\sigma$ (in a semi-active way) is

$$t_{\sigma,i} := \begin{cases} \max(r_i, t_{\sigma,j} + p_j), & \text{if } \sigma(i) > 1, \\ r_i, & \text{if } \sigma(i) = 1, \end{cases} \tag{1}$$

where $j \in N$ such that $\sigma(j) = \sigma(i) - 1$.

Hence, the completion time $C(\sigma, i)$ of the job of agent $i$ with respect to $\sigma$ is equal to $t_{\sigma,i} + p_i$. The total costs $c_\sigma(S)$ of a coalition $S \subset N$ is given by

$$c_\sigma(S) := \sum_{i \in S} \alpha_i(C(\sigma, i)) + \beta_i.$$

The (maximal) cost savings of a coalition $S$ depend on the set of admissible rearrangements of this coalition. We call a bijection $\sigma : N \to \{1, \dots, n\}$ *admissible for* $S$ if it satisfies the following two conditions:

(i) the starting time of each agent outside the coalition $S$ is equal to his starting time in the initial order: $t_{\sigma_0,i} = t_{\sigma,i}$ for all $i \in N \backslash S$;

(ii) the agents of $S$ are not allowed to jump over players outside $S$:

$$P(i) \cap N \backslash S = P(\sigma, i) \cap N \backslash S, \quad \text{for all } i \in S.$$

The set of admissible rearrangements for a coalition $S$ is denoted by $\Sigma_S$.

Before formally introducing sequencing games, we recall some well-known facts concerning cooperative games.

A *cooperative game* is a pair $(N, v)$ where $N$ is a finite set of players and $v$ is a mapping $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ and $2^N$ denoting the collection of all subsets of $N$.

A game $(N, v)$ is called *convex* if for all coalitions $S, T \in 2^N$ it holds that

$$v(S \cup T) + v(S \cap T) \geqslant v(S) + v(T), \tag{2}$$

or, equivalently, if for all coalitions $S, T \in 2^N$ and all $i \in N$ with $S \subset T \subset N \backslash \{i\}$ it holds that

$$v(T \cup \{i\}) - v(T) \geqslant v(S \cup \{i\}) - v(S).$$

Cooperative game theory focuses on 'fair' and/or 'stable' division rules for the value $v(N)$ of the grand coalition. A core element $x = (x_i)_{i \in N} \in \mathbb{R}^N$ is such that no coalition has an incentive to split of, i.e.,

$$\sum_{i\in N} x_i = v(N) \quad \text{and} \quad \sum_{i\in S} x_i \geqslant v(S), \quad \text{for all } S \in 2^N.$$

The core $C(v)$ consists of all core elements. A game is called *balanced* if its core is nonempty. We note that convex games are balanced.

Given a sequencing situation $(\sigma_0, r, p, \alpha)$, the corresponding sequencing game [2] is defined in such a way that the worth of a coalition $S$ is equal to the maximal cost savings the coalition can achieve by means of an admissible rearrangement. Formally we have

$$\begin{aligned} v(S) &= \max_{\sigma\in\Sigma_S} \Big\{ \sum_{i\in S} (\alpha_i C(\sigma_0, i) + \beta_i) - \sum_{i\in S} (\alpha_i C(\sigma, i) + \beta_i) \Big\} \\ &= \max_{\sigma\in\Sigma_S} \Big\{ \sum_{i\in S} \alpha_i [C(\sigma_0, i) - C(\sigma, i)] \Big\}. \end{aligned} \tag{3}$$

A coalition $S$ is called *connected* with respect to $\sigma_0$ if for all $i, j \in S$ and $k \in N$, $\sigma_0(i) < \sigma_0(k) < \sigma_0(j)$ implies $k \in S$. A connected coalition $S \subset T$ is a *component* of $T$ if $S \cup \{i\}$ is not connected for every $i \in T\backslash S$. The components of $T$ form a partition of $T$, denoted by $T/\sigma_0$. According to condition (ii) of an admissible rearrangement of a coalition $S$ the players of $S$ are not allowed to jump over players outside the coalition. This implies that an optimal rearrangement is such that the players in each component are rearranged optimally. Hence, for any coalition $T$,

$$v(T) = \sum_{S\in T/\sigma_0} v(S). \tag{4}$$

In the final part of this section we consider sequencing games that arise from sequencing situations with criteria equivalent to the weighted cost criterion that is used in this paper. Criteria are called *equivalent* if any optimal rearrangement with respect to one criterion is also an optimal rearrangement for the others. In several textbooks (cf. [1]) it is shown that the weighted flow time criterion, the weighted waiting time criterion and the weighted lateness criterion are equivalent to the weighted cost criterion. With respect to the corresponding sequencing games one easily verifies the following proposition.

**Proposition 1.** *Each sequencing situation with a criterion equivalent to the weighted cost criterion generates the sequencing game described in* (3).

## 3. On the convexity of sequencing games

Curiel et al. [2] showed that each sequencing game arising from a sequencing situation with zero ready times is convex. In this section we concentrate on sequencing situations $(\sigma_0, r, p, \alpha)$ in which all jobs have equal processing time and the ready times of each job are multiples of the processing time and show that the corresponding games are convex. Without loss of generality we restrict attention to sequencing situations $(\sigma_0, r, p, \alpha)$ with

(A2) $\quad r_i \in \mathbb{N}$ and $p_i = 1, \quad$ for all $i \in N$.

Further, we assume that there are no time gaps in the job processing according to the initial order $\sigma_0$, i.e.,

(A3) $\quad t_{\sigma_0,j} = t_{\sigma_0,i} + p_i, \quad$ for all $i, j \in N$ with $\sigma_0(j) = \sigma_0(i) + 1$.

Note that (A3) does not restrict the convexity result of this section. For, suppose that in the processing of the jobs in $N$ according to $\sigma_0$ we have the clusters $N_1, N_2, \ldots, N_s$ with $\bigcup_{k=1}^{s} N_k = N$, in such a way that each cluster is being processed without time gaps in-between, and that there is a time gap between consecutive clusters, then one readily verifies from the fact that $r_i \leqslant r_j$ for all $i, j \in N$ with $\sigma_0(i) < \sigma_0(j)$ (cf. (A1)) that for all $S \subset N$ it holds that

$$v(S) = \sum_{k=1}^{s} v(N_k \cap S).$$

Hence, to show convexity of $v$, it suffices to prove that the restricted games $(N_k, v_{|N_k})$, with $v_{|N_k}(S) = v(S)$ for all $S \subset N_k$, are convex for all $k \in \{1, \ldots, s\}$.

Finally, for convenience, we rescale time in the sense that we assume

(A4) $\quad r_i = 0, \quad$ for $i \in N$ with $\sigma_0(i) = 1$.

Combining (A2)–(A4), we find

$$t_{\sigma_0,i} = \sigma_0(i) - 1, \quad \text{for all } i \in N. \tag{5}$$

To calculate the worth of a coalition $S$ in the corresponding sequencing game, we need to find an optimal (= cost minimizing) rearrangement $\hat{\sigma}_S$ in the set of admissible rearrangements of the coalition $S$. For this we use the following procedure [6], which generalizes the Smith rule [9], and which can be applied for all sequencing situations that satisfy (A2). To obtain the optimal order of the coalition $N$, one inductively considers at each possible starting time $t \in \{0, 1, \ldots\}$ all jobs that are available at time moment $t$, i.e., the jobs that are not processed before $t$ and that have a ready time smaller than or equal to $t$. If job $i$ has the highest urgency $\alpha_i$ (cf. [9]) among all available jobs at $t$, then $i$ will be processed at that time. If there is more than one available job at time $t$ with the highest urgency, we pick the one with the smallest index number. Note that (A3) and (A4) imply that the sets of available jobs at $t \in \{0, \ldots, n-1\}$ are never empty. So, the procedure will stop after $n$ steps.

A similar procedure can be applied to find an optimal rearrangement $\hat{\sigma}_S$ for a connected coalition $S$. By definition of admissible rearrangements we have $t_{\hat{\sigma}_S,j} = t_{\sigma_0,j}$ for all $j \in N \backslash S$. For the possible starting times $t \in \{0, 1, 2, \ldots\} \backslash \{t_{\sigma_0,j}\}_{j \in N \backslash S}$ one (inductively) considers those jobs in $S$ that are available at time $t$. By $A_t(S)$ we denote the set of available jobs at time $t$ in determining the optimal rearrangement with respect to $S$.

From the jobs in $A_t(S)$ the job with the highest urgency (and lowest index) will start processing at time $t$. Again (A3) and (A4) imply that $A_t(S) \neq \emptyset$ at each time moment $t \in \{0,1,2,\ldots,n-1\} \backslash \{t_{\sigma_0,j}\}_{j \in N \backslash S}$ and so the procedure stops after $|S|$ steps.

The optimal rearrangement of an arbitrary coalition $S$ can be found by 'combining' the optimal rearrangements of the components of $S$ in the obvious way. Further, note that the above procedure implies that there are no time gaps in the processing of $N$ according to an optimal rearrangement $\hat{\sigma}_S$, and therefore

$$t_{\hat{\sigma}_S,i} = \hat{\sigma}_S(i) - 1, \quad \text{for all } i \in N, \tag{6}$$

for all $S \in 2^N \backslash \{\emptyset\}$, and moreover we have

$$t_{\hat{\sigma}_S,i} = t_{\sigma_0,j}, \quad \text{if } \hat{\sigma}_S(i) = \sigma_0(j).$$

To illustrate the procedure we give the following example. For notational convenience we denote a bijection $\sigma : N \to \{1,\ldots,n\}$ by an $n$-dimensional vector $(i_1,\ldots,i_n)$ with $\{i_1,\ldots,i_n\} = N$ and where $i_k$ denotes the player that is assigned to position $k$.

**Example 2.** Let $N = \{1,2,3,4\}$, $\sigma_0 = (1,2,3,4)$, $r = (0,0,1,1)$, $p = (1,1,1,1)$ and $\alpha = (1,2,3,4)$. Note that (A1)–(A4) are satisfied. First we determine the optimal rearrangement of $N$. Let $A_t(N)$ be the set of jobs available at time $t \in \{0,1,2,\ldots\}$. Then $A_0(N) = \{1,2\}$ and, since $\alpha_2 > \alpha_1$, we have $t_{\hat{\sigma}_N,2} = 0$. Similarly, $A_1(N) = \{1,3,4\}$ and $\alpha_4 > \alpha_3 > \alpha_1$ implies $t_{\hat{\sigma}_N,4} = 1$. Since $A_2(N) = \{1,3\}$ and $\alpha_3 > \alpha_1$, $t_{\hat{\sigma}_N,3} = 2$. Finally $A_3(N) = \{1\}$ and $t_{\hat{\sigma}_N,1} = 3$. We may conclude that $\hat{\sigma}_N = (2,3,4,1)$.

Secondly we consider $S = \{1,2,4\}$. By definition, $t_{\hat{\sigma}_S,3} = t_{\sigma_0,3} = \sigma_0(3) - 1 = 2$. The components of $S$ are $\{1,2\}$ and $\{4\}$. Then $A_0(\{1,2\}) = \{1,2\}$. Since $\alpha_2 > \alpha_1$, we find $t_{\hat{\sigma}_S,2} = 0$. Further $A_1(\{1,2\}) = \{1\}$ and we have $t_{\hat{\sigma}_S,1} = 1$. Further, $A_3(\{4\}) = \{4\}$. Consequently, we have $t_{\hat{\sigma}_S,4} = 3$. Summarizing, $\hat{\sigma}_S = (2,1,3,4)$.

As a consequence of the above procedure we are able to rewrite the expression (3) which defines the value of a coalition $S$. Let $\hat{\sigma}_S$ be the optimal rearrangement of $S$ obtained by the described procedure; then,

$$\begin{aligned} v(S) &= \sum_{i \in S} \alpha_i \left[ C(\sigma_0,i) - C(\hat{\sigma}_S,i) \right] = \sum_{i \in S} \alpha_i \left[ (t_{\sigma_0,i}+1) - (t_{\hat{\sigma}_S,i}+1) \right] \\ &= \sum_{i \in S} \alpha_i \left[ \sigma_0(i) - \hat{\sigma}_S(i) \right], \end{aligned} \tag{7}$$

where the first equality follows from substituting $\hat{\sigma}_S$ in (3), the second equality follows from (A2), and the last one from (5) and (6).

The next lemma shows that the restriction $(S, v_{|S})$ of a sequencing game $(N, v)$, arising from a sequencing situation that satisfies (A1)–(A4), is again such a sequencing game if the coalition $S$ is connected. Since the proof of this lemma is rather technical but straightforward, it is omitted.

**Lemma 3.** *Let $(\sigma_0, r, p, \alpha)$ be a sequencing situation satisfying* (A1)–(A4) *and let $(N, v)$ be the corresponding sequencing game. If the coalition $S$ is connected with respect to $\sigma_0$, then the game $(S, v_{|S})$ is the sequencing game corresponding to the sequencing situation*

$$(\overline{\sigma_0}, (\overline{r_i})_{i\in S}, (p_i)_{i\in S}, (\alpha_i)_{i\in S}),$$

*where the bijection $\overline{\sigma_0} : S \to \{1, \dots, |S|\}$ is defined by $\overline{\sigma_0}(i) = \sigma_0(i) + 1 - \min_{j\in S} \sigma_0(j)$ for all $i \in S$ and $\overline{r}_i = \max\{r_i - \min_{j\in S} t_{\sigma_0, j}, 0\}$ for all $i \in S$.*

Now we can formulate the following theorem.

**Theorem 4.** *Let $(\sigma_0, r, p, \alpha)$ be a sequencing situation satisfying* (A1)–(A4) *and let $(N, v)$ be the corresponding sequencing game. Then $(N, v)$ is convex.*

**Proof.** The proof is by induction on the number of players. Obviously, if $|N| = 2$, the sequencing game $(N, v)$ is convex because $v(N) \geqslant 0$ and $v(\{i\}) = 0$ for all $i \in N$. Let $(N, v)$ be a sequencing game with $|N| \geqslant 3$. Let $|N| = n$ and assume that the theorem holds for any sequencing game $(M, w)$ arising from a sequencing situation that satisfies (A1)–(A4), with $2 \leqslant |M| < n$. Let $i \in N$ and $S, T \in 2^N$ be such that $S \subset T \subset N\backslash\{i\}$. We have to prove that

$$v(T \cup \{i\}) - v(T) \geqslant v(S \cup \{i\}) - v(S). \tag{8}$$

(a) Suppose there exists a player $j \in N$, $j \neq i$, such that $j \notin T$. We can unambiguously determine $T_1 \subset T$ and $T_2 \subset T$ such that $T_1 \cup \{i\} \cup T_2$ is a component of $T \cup \{i\}$ and $T_i \neq \emptyset$, $i = 1, 2$, implies that $T_i$ is a component of $T$. Consequently,

$$v(T \cup \{i\}) - v(T) = v(T_1 \cup \{i\} \cup T_2) - v(T_1) - v(T_2).$$

Similarly, let $S_1 \subset S$ and $S_2 \subset S$ be the sets such that $S_1 \cup \{i\} \cup S_2$ is a component of $S \cup \{i\}$ and $S_i \neq \emptyset$, $i = 1, 2$, implies that $S_i$ is a component of $S$. Note that without loss of generality we may assume that $S_1 \subset T_1$ and $S_2 \subset T_2$. We can write

$$v(S \cup \{i\}) - v(S) = v(S_1 \cup \{i\} \cup S_2) - v(S_1) - v(S_2),$$

and therefore it suffices to show that

$$v(T_1 \cup \{i\} \cup T_2) - v(T_1 \cup T_2) \geqslant v(S_1 \cup \{i\} \cup S_2) - v(S_1 \cup S_2). \tag{9}$$

Since $T_1 \cup \{i\} \cup T_2$ is a connected subset of $N\backslash\{j\}$, Lemma 3 and the induction hypothesis imply that $(T_1 \cup \{i\} \cup T_2, v_{|T_1 \cup \{i\} \cup T_2})$ is a convex sequencing game and therefore (9) is satisfied.

(b) Hence, we may assume that $T = N\backslash\{i\}$. Moreover, (a) also implies that it is sufficient to prove

$$v(N) - v(N\backslash\{i\}) \geqslant v(N\backslash\{j\}) - v(N\backslash\{i, j\}), \quad \text{for all } j \in N,\ i \neq j.$$

Let $j \in N$. Without loss of generality we assume that $\sigma_0(i) < \sigma_0(j)$, otherwise we can interchange the role of $i$ and $j$. Since

$$\begin{aligned}
&v(N) - v(N\backslash\{i\}) - v(N\backslash\{j\}) + v(N\backslash\{i,j\}) \\
&\quad = v(N) - (v(P(i)) + v(F(i))) - (v(P(j)) + v(F(j))) \\
&\qquad + v(P(i)) + v(F(j)) + v(F(i) \cap P(j)) \\
&\quad = v(N) - v(P(j)) + v(F(i) \cap P(j)) - v(F(i)) \\
&\quad = [v(N\backslash\{n_0\}) + v(P(j) \cap F(i)) - v(F(i)\backslash\{n_0\}) - v(P(j))] \\
&\qquad + [v(N) - v(N\backslash\{n_0\}) + v(F(i)\backslash\{n_0\}) - v(F(i))],
\end{aligned}$$

where $n_0 := \sigma_0^{-1}(n)$, it suffices to show that

$$v(N\backslash\{n_0\}) + v(P(j) \cap F(i)) - v(F(i)\backslash\{n_0\}) - v(P(j)) \geqslant 0 \tag{10}$$

and

$$v(N) - v(N\backslash\{n_0\}) + v(F(i)\backslash\{n_0\}) - v(F(i)) \geqslant 0. \tag{11}$$

Inequality (10) follows from the fact that both $F(i)\backslash\{n_0\}$ and $P(j)$ are contained in $N\backslash\{n_0\}$ and that, by induction, the game $(N\backslash\{n_0\}, v_{|N\backslash\{n_0\}})$ is convex.

With respect to (11) we have

$$\begin{aligned}
&v(N) - v(N\backslash\{n_0\}) + v(F(i)\backslash\{n_0\}) - v(F(i)) \\
&= \sum_{l \in F(\hat{\sigma}_N, n_0)} (\alpha_{n_0} - \alpha_l) - \sum_{k \in F(\hat{\sigma}_{F(i)}, n_0)} (\alpha_{n_0} - \alpha_k) \\
&= \sum_{l:\, \hat{\sigma}_N(n_0) < \hat{\sigma}_N(l) \leqslant \hat{\sigma}_{F(i)}(n_0)} (\alpha_{n_0} - \alpha_l) + \sum_{l:\, \hat{\sigma}_N(l) > \hat{\sigma}_{F(i)}(n_0)} (\alpha_{n_0} - \alpha_l) \\
&\quad - \sum_{k:\, \hat{\sigma}_{F(i)}(k) > \hat{\sigma}_{F(i)}(n_0)} (\alpha_{n_0} - \alpha_k) \\
&= \sum_{l:\, \hat{\sigma}_N(n_0) < \hat{\sigma}_N(l) \leqslant \hat{\sigma}_{F(i)}(n_0)} (\alpha_{n_0} - \alpha_l) \\
&\quad - \sum_{l:\, \hat{\sigma}_N(l) > \hat{\sigma}_{F(i)}(n_0)} \alpha_l + \sum_{k:\, \hat{\sigma}_{F(i)}(k) > \hat{\sigma}_{F(i)}(n_0)} \alpha_k \geqslant 0,
\end{aligned}$$

where the first equality is proved in detail in Lemma A.2 of the Appendix (in particular it uses (7)), the second equality uses the fact that $\hat{\sigma}_{F(i)}(n_0) \geqslant \hat{\sigma}_N(n_0)$ which is shown in Lemma A.3 of the Appendix and the inequality follows from the Lemmata A.4 and A.5 of the Appendix. □

The following example shows that general sequencing games with ready times are not necessarily convex. Nevertheless, each sequencing game has a nonempty core: Curiel et al. [3] consider a class of balanced games that contains all sequencing games with ready times.

**Example 5.** Let $N = \{1,2,3\}$, $\sigma_0 = (1,2,3)$, $r = (0,0,1)$, $p = (1,2,3)$ and $\alpha = (1,3,12)$. The costs according to the initial order equals 82. The optimal rearrangement $\hat{\sigma}_N$ equals $(1,3,2)$ with corresponding costs of 67. Consequently, we have that $v(N) = 15$. Further, $v(\{2,3\}) = 15$ since $\hat{\sigma}_{\{2,3\}} = (1,3,2)$ and $v(\{1,2\}) = 1$ since $\hat{\sigma}_{\{1,2\}} = (2,1,3)$. From this we can infer that $(N,v)$ is not a convex game:

$$v(N) - v(\{2,3\}) = 0 < 1 = v(\{1,2\}) - v(\{2\}).$$

The convexity of sequencing games is an open problem in case we restrict attention to the special class of sequencing situations in which all jobs have equal processing times and the ready times are integers.

## 4. Marginals and solution concepts

Let $(N,v)$ be a game in coalitional form and let $\Pi_N$ be the set of all bijections of $N$ to $\{1,\dots,n\}$. Then the $k$th coordinate of the marginal vector $m^\pi(v)$, $\pi \in \Pi_N$, is defined by

$$m_k^\pi(v) := v(\{j \in N \mid \pi(j) \leqslant \pi(k)\}) - v(\{j \in N \mid \pi(j) < \pi(k)\}).$$

Shapley [8] and Ichiishi [5] (cf. [4]) showed that the marginal vectors are the extreme points of the core if and only if the game is convex. In particular, this implies that the structure of the core of a sequencing games arising from sequencing situations that satisfy (A1)–(A4) can be studied by looking at the marginals.

Moreover, for sequencing games we can distinguish between two types of marginals: connected and disconnected marginals. A marginal $m^\pi(v)$ is called *connected* if for each $k \in N$ the set $\{j \mid \pi(j) \leqslant \pi(k)\}$ is connected with respect to $\sigma_0$, and *disconnected* otherwise. For example, let $N = \{1,2,3\}$, $\sigma_0 = (1,2,3)$, $r = (0,0,1)$, $p = (1,1,1)$ and $\alpha = (1,3,12)$. Then the connected marginals are

$$(v(\{1\}), v(\{1,2\}) - v(\{2\}), v(N) - v(\{1,2\})) = (0,2,11),$$

$$(v(\{1,2\}) - v(\{2\}), v(\{2\}), v(N) - v(\{1,2\})) = (2,0,11),$$

$$(v(N) - v(\{2,3\}), v(\{2\}), v(\{2,3\}) - v(\{2\})) = (4,0,9),$$

$$(v(N) - v(\{2,3\}), v(\{2,3\}) - v(\{3\}), v(\{3\})) = (4,9,0);$$

and the disconnected marginals are the remaining ones:

$$(v(\{1\}), v(N) - v(\{1,3\}), v(\{1,3\}) - v(\{1\})) = (0,13,0),$$

$$(v(\{1,3\}) - v(\{3\}), v(N) - v(\{1,3\}), v(\{3\})) = (0,13,0).$$

By definition, the Shapley value [7] of $(N,v)$ is the average of all marginals: $\phi(v) = (1/n!)\sum_{\pi\in\Pi_N} m^{\pi}(v)$. In our example, $\phi(v) = \frac{1}{6}(10,37,31)$. Note that by convexity of the sequencing game the Shapley value corresponds to the barycentre of the core. Another interesting element of the core is the average $\gamma(v)$ of all connected marginals. In the example, $\gamma(v) = \frac{1}{4}(10,11,31)$.

Clearly, there are at most $2^{n-1}$ distinct connected marginals. Further, it is not difficult to verify that there are at most $(n-1)!\frac{1}{2}(n-2)$ distinct disconnected marginals. Since $2^{n-1}+(n-1)!\frac{1}{2}(n-2) < (n-1)!+(n-1)!\frac{1}{2}(n-2) = \frac{1}{2}(n!)$ for $n \geqslant 4$, the number of extreme points of the core of a convex sequencing game is less than $\frac{1}{2}(n!)$.

## Appendix

Let $n_0 := \sigma_0^{-1}(n)$ be the player in the last position with respect to the initial order. The following lemma shows that the optimal rearrangement $\hat{\sigma}_N$ of $N$ is obtained by 'inserting' player $n_0$ in the optimal rearrangement $\hat{\sigma}_{N\setminus\{n_0\}}$ of $N\setminus\{n_0\}$.

**Lemma A.1.** *Let $(\sigma_0, r, p, \alpha)$ be a sequencing situation satisfying* (A1)–(A4). *Then, $\hat{\sigma}_{N\setminus\{n_0\}}(k) = \hat{\sigma}_N(k)$ if $\hat{\sigma}_N(k) < \hat{\sigma}_N(n_0)$ and $\hat{\sigma}_{N\setminus\{n_0\}}(k) = \hat{\sigma}_N(k) - 1$ if $\hat{\sigma}_N(k) > \hat{\sigma}_N(n_0)$.*

**Proof.** Let $A_t(N\setminus\{n_0\})$, $t\in\{0,1,\dots,n-2\}$, and $A_t(N)$, $t\in\{0,1,\dots,n-1\}$, be the nonempty sets of available jobs within $N\setminus\{n_0\}$ and $N$, respectively, at time $t$. Clearly, for $t\in\{0,1,\dots,r_{n_0}-1\}$, an inductive argument shows that $A_t(N\setminus\{n_0\}) = A_t(N)$ and, consequently, $t_{\hat{\sigma}_{N\setminus\{n_0\}},k} = t$ if and only if $t_{\hat{\sigma}_N,k} = t$.

If $r_{n_0} = n-1$, the lemma follows. Assume that $r_{n_0} < n-1$. Since (A1) and the above reasoning imply that all remaining jobs are available at $t = r_{n_0}$, or more precisely, that

$$N\setminus\{j\in N \mid t_{\hat{\sigma}_N,j} \leqslant r_{n_0} - 1\} = A_{r_{n_0}}(N) = A_{r_{n_0}}(N\setminus\{n_0\}) \cup \{n_0\},$$

it readily follows (inductively) that

$$A_t(N\setminus\{n_0\}) = \begin{cases} A_t(N)\setminus\{n_0\}, & \text{if } r_{n_0} \leqslant t \leqslant t_{\hat{\sigma}_N,n_0} - 1, \\ A_{t+1}(N), & \text{if } t_{\hat{\sigma}_N,n_0} \leqslant t \leqslant n-2, \end{cases}$$

which finishes the proof. □

Let $\sigma_0 = (i_1,\dots,i_n)$ and let $S = \{i_r,\dots,i_{s+1}\}$ be a connected set. From Lemma A.1 it immediately follows that the optimal rearrangement of $S$ is obtained by inserting player $i_{s+1}$ in the optimal rearrangement of the coalition $\{i_r,\dots,i_s\}$.

Lemma A.2 gives an expression for the difference of the values of a tail $S$ and $S\setminus\{n_0\}$. Here $S$ is called a tail if $S$ is connected and $n_0\in S$.

**Lemma A.2.** *Let $(\sigma_0, r, p, \alpha)$ be a sequencing situation satisfying* (A1)–(A4) *and let $(N,v)$ be the corresponding sequencing game. Let $S$ be connected with $n_0\in S$. Then,*

$$v(S)-v(S\backslash\{n_0\}) = \sum_{k\in F(\hat{\sigma}_S,n_0)} (\alpha_{n_0}-\alpha_k).$$

**Proof.**

$$\begin{aligned}
&v(S)-v(S\backslash\{n_0\})\\
&\quad=\sum_{i\in S}\alpha_i[\sigma_0(i)-\hat{\sigma}_S(i)]-\sum_{i\in S\backslash\{n_0\}}\alpha_i[\sigma_0(i)-\hat{\sigma}_{S\backslash\{n_0\}}(i)]\\
&\quad=\alpha_{n_0}(n-\hat{\sigma}_S(n_0))+\sum_{i\in S\backslash\{n_0\}}\alpha_i[\sigma_0(i)-\hat{\sigma}_S(i)]\\
&\qquad-\sum_{i\in S\backslash\{n_0\}}\alpha_i[\sigma_0(i)-\hat{\sigma}_{S\backslash\{n_0\}}(i)]\\
&\quad=\alpha_{n_0}(n-\hat{\sigma}_S(n_0))-\sum_{i\in F(\hat{\sigma}_S,n_0)}\alpha_i=\sum_{i\in F(\hat{\sigma}_S,n_0)}(\alpha_{n_0}-\alpha_i).
\end{aligned}$$

The first equality follows by (7), the third equality is a consequence of Lemma A.1 and the last equality holds since $n-\hat{\sigma}_S(n_0)=|F(\hat{\sigma}_S,n_0)|$. □

The following lemma shows that in the optimal rearrangement of $N$ the position of player $n_0$ is smaller than or equal to the position of player $n_0$ in the optimal rearrangement of any other tail.

**Lemma A.3.** *Let* $(\sigma_0,r,p,\alpha)$ *be a sequencing situation satisfying* (A1)–(A4). *Then,* $\hat{\sigma}_N(n_0)\leqslant\hat{\sigma}_{F(i)}(n_0)$ *for all* $i\in N\backslash\{n_0\}$.

**Proof.** Let $i\in N\backslash\{n_0\}$. Suppose $\hat{\sigma}_N(n_0)>\hat{\sigma}_{F(i)}(n_0)$. Choose $n_1\in N\backslash\{n_0\}$ such that $\hat{\sigma}_N(n_1)=\hat{\sigma}_{F(i)}(n_0)$. Since $r_{n_0}\leqslant t_{\hat{\sigma}_{F(i)},n_0}=t_{\hat{\sigma}_N,n_1}$ the optimality of $\hat{\sigma}_N$ implies that $\alpha_{n_1}\geqslant\alpha_{n_0}$.

Suppose $n_1\in P(i)\cup\{i\}$. Then,

$$\begin{aligned}
\hat{\sigma}_N(n_1)&=|P(\hat{\sigma}_N,n_1)\cap F(i)|+|P(\hat{\sigma}_N,n_1)\cap(P(i)\cup\{i\})|+1\\
&\leqslant|P(\hat{\sigma}_N,n_1)\cap F(i)|+|P(i)\cup\{i\}|\\
&=|P(\hat{\sigma}_N,n_1)\cap F(i)|+\hat{\sigma}_{F(i)}(i).
\end{aligned}\tag{12}$$

The inequality follows from $(P(\hat{\sigma}_N,n_1)\cap(P(i)\cup\{i\}))\subsetneq P(i)\cup\{i\}$ which follows from the fact that $n_1\notin P(\hat{\sigma}_N,n_1)$. We also have

$$\begin{aligned}
\hat{\sigma}_{F(i)}(n_0)&=|P(\hat{\sigma}_{F(i)},n_0)\cap F(i)|+|P(i)\cup\{i\}|+1\\
&>|P(\hat{\sigma}_{F(i)},n_0)\cap F(i)|+|P(i)\cup\{i\}|\\
&\geqslant|P(\hat{\sigma}_N,n_1)\cap F(i)|+\hat{\sigma}_{F(i)}(i).
\end{aligned}\tag{13}$$

The last inequality follows from the fact that $(P(\hat{\sigma}_N,n_1)\cap F(i))\subset(P(\hat{\sigma}_{F(i)},n_0)\cap F(i))$. For, let $k\in P(\hat{\sigma}_N,n_1)\cap F(i)$. Since $n_1\in P(i)\cup\{i\}$ we have by (A1) that $r_{n_1}\leqslant r_k$. This implies that $\alpha_k>\alpha_{n_1}$. Since $\alpha_{n_1}\geqslant\alpha_{n_0}$, this implies $k\in P(\hat{\sigma}_{F(i)},n_0)\cap F(i)$.

However, from inequalities (12) and (13) it follows that

$$\hat{\sigma}_N(n_1) \leqslant |P(\hat{\sigma}_N, n_1) \cap F(i)| + \hat{\sigma}_{F(i)}(i) < \hat{\sigma}_{F(i)}(n_0),$$

which contradicts the fact that $\hat{\sigma}_N(n_1) = \hat{\sigma}_{F(i)}(n_0)$.

So we may assume that $n_1 \in F(i)$. Together with $\alpha_{n_1} \geqslant \alpha_{n_0}$, this yields $n_1 \in P(\hat{\sigma}_{F(i)}, n_0)$ and consequently, $\hat{\sigma}_{F(i)}(n_0) > \sigma_0(i) + 1$. Now choose $n_2 \in N \backslash \{n_0, n_1\}$ such that $\hat{\sigma}_N(n_2) = \hat{\sigma}_{F(i)}(n_1)$. The optimality of $\hat{\sigma}_N$ implies $\alpha_{n_2} \geqslant \alpha_{n_1}$.

Suppose $n_2 \in P(i) \cup \{i\}$. An analogous reasoning as above leads to $\hat{\sigma}_N(n_2) \leqslant |P(\hat{\sigma}_N, n_2) \cap F(i)| + \hat{\sigma}_{F(i)}(i)$ and $\hat{\sigma}_{F(i)}(n_1) > |P(\hat{\sigma}_N, n_2) \cap F(i)| + \hat{\sigma}_{F(i)}(i)$ and we again have a contradiction.

So we may assume that $n_2 \in F(i)$. Then $n_2 \in P(\hat{\sigma}_{F(i)}, n_1)$ and we have a contradiction if $\hat{\sigma}_{F(i)}(n_0) = \hat{\sigma}_{F(i)}(i) + 2$. Hence, $\hat{\sigma}_{F(i)}(n_0) > \hat{\sigma}_{F(i)}(i) + 2$. Now choose $n_3 \in N \backslash \{n_0, n_1, n_2\}$ such that $\hat{\sigma}_N(n_3) = \hat{\sigma}_{F(i)}(n_2)$.

Using the same line of argument as above we then find that $\hat{\sigma}_{F(i)}(n_0) > \hat{\sigma}_{F(i)}(i) + 3$. We may conclude that if $\hat{\sigma}_{F(i)}(n_0) = \hat{\sigma}_{F(i)}(i) + k$, we arrive at a contradiction after $k$ repetitions. □

The next lemma shows that in the optimal rearrangement of any tail the jobs from $n_0$ on are ordered in decreasing urgency.

**Lemma A.4.** *Let $(\sigma_0, r, p, \alpha)$ be a sequencing situation satisfying* (A1)–(A4). *Let $S$ be connected with $n_0 \in S$ and let $k, l \in N$ be such that $\hat{\sigma}_S(n_0) \leqslant \hat{\sigma}_S(k) < \hat{\sigma}_S(l)$. Then $\alpha_k \geqslant \alpha_l$.*

**Proof.** Since $r_{n_0} \leqslant t_{\hat{\sigma}_S, n_0}$ and by (A1) we have $r_i \leqslant r_{n_0}$ for all $i \in N$ it holds that $l \in A_{t_{\hat{\sigma}_S,k}}(S)$. Hence, $\alpha_k \geqslant \alpha_l$. □

**Lemma A.5.** *Let $(\sigma_0, r, p, \alpha)$ be a sequencing situation satisfying* (A1)–(A4). *Let $i \in N \backslash \{n_0\}$, $k \in F(i)$ and $l \in N$ be such that $\hat{\sigma}_{F(i)}(k) \geqslant \hat{\sigma}_{F(i)}(n_0)$ and $\hat{\sigma}_{F(i)}(k) = \hat{\sigma}_N(l)$. Then $\alpha_k \geqslant \alpha_l$.*

**Proof.** The proof is by induction on the number of players. If $|N| = 2$, then $\sigma_0 = (i, n_0)$ and so $F(i) = \{n_0\}$, $k = n_0$ and $\hat{\sigma}_{F(i)} = \sigma_0$. In case $\hat{\sigma}_N = \sigma_0$, then $l = n_0$ and so $\alpha_{n_0} = \alpha_l$. In case $\hat{\sigma}_N = (n_0, i)$, then $l = i$ and $\alpha_{n_0} \geqslant \alpha_l$ due to the algorithm for the determination of the optimal rearrangement $\hat{\sigma}_N$.

Let $(\sigma_0, r, p, \alpha)$ be a sequencing situation with $\hat{\sigma}_{F(i)}(k) = \hat{\sigma}_{F(i)}(n_0)$ and $\hat{\sigma}_{F(i)}(k) = \hat{\sigma}_N(l)$ for some $i \in N \backslash \{n_0\}$, $k \in F(i)$ and $l \in N$ where $n := |N| \geqslant 3$. By induction we may assume that the lemma holds for any sequencing situation with less than $n$ players. We distinguish two cases.

(i) $\hat{\sigma}_{F(i)}(n_0) = \hat{\sigma}_{F(i)}(k)$, i.e., $k = n_0$. From Lemma A.3 it follows that $\hat{\sigma}_N(n_0) \leqslant \hat{\sigma}_{F(i)}(n_0) = \hat{\sigma}_N(l)$. Then Lemma A.4 implies that $\alpha_{n_0} \geqslant \alpha_l$.

(ii) $\hat{\sigma}_{F(i)}(n_0) < \hat{\sigma}_{F(i)}(k)$. Using Lemma A.3, we have $\hat{\sigma}_N(n_0) \leqslant \hat{\sigma}_{F(i)}(n_0) < \hat{\sigma}_{F(i)}(k) = \hat{\sigma}_N(l)$. By applying Lemma A.1 to $\hat{\sigma}_N$ and $\hat{\sigma}_{F(i)}$, it follows that $\hat{\sigma}_{N \backslash \{n_0\}}(l) =$

$\hat{\sigma}_N(l) - 1$ and $\hat{\sigma}_{F(i)\backslash\{n_0\}}(k) = \hat{\sigma}_{F(i)}(k) - 1$. Therefore, $k \in F(i)\backslash\{n_0\}$ and $l \in F(i)\backslash\{n_0\}$ satisfy $\hat{\sigma}_{F(i)\backslash\{n_0\}}(k) = \hat{\sigma}_{N\backslash\{n_0\}}(l)$. From the induction hypotheses applied to the corresponding sequencing situation with player set $N\backslash\{n_0\}$ it follows that $\alpha_k \geqslant \alpha_l$. $\square$

## Acknowledgements

The authors wish to thank two anonymous referees of Mathematical Programming for their valuable comments and suggestions.

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